Inequality
https://www.linkedin.com/groups/8313943/8313943-6409422411351101442
Let x,y and z be real numbers such that $x,y,z \geq 1$ and $x+y+z=6$, prove that
$(x^2+2)(y^2+2)(z^2+2) \leq 216.$

Solution by Arkady Alt , San Jose, California, USA.

Let $p := xy+yz+zx, q := xyz$. Then $x^2+y^2+z^2 = 36-2p, x^2y^2+y^2z^2+z^2x^2 = p^2-12q$ and $(x^2+2)(y^2+2)(z^2+2) = x^2y^2z^2+2(x^2y^2+y^2z^2+z^2x^2)+4(x^2+y^2+z^2)+8 =$
$q^2+2(p^2-12q)+4(36-2p)+8 = (12-q)^2+2(p-2)^2.$

Since $q = xyz \leq \left(\frac{x+y+z}{3}\right)^3 = 8$ and* $9q = 9xyz \geq 4(x+y+z)(xy+yz+zx) -$
$(x+y+z)^3 = 24p-6^3 \Leftrightarrow 3q \geq 8p-72.$ (Schure Inequality $\sum x(x-y)(x-z) \geq 0$).

Also we will prove that $q \geq 4$.

Indeed, assuming $x \leq y \leq z$ we obtain $1 \leq x \leq y \leq 6-x-y \Leftrightarrow$
$1 \leq x \leq 2$ & $x \leq y \leq 3-\frac{x}{2}$ and $xyz = xy(6-x-y) = x(-y^2+y(6-x)).$

Since for $y \in \left[x, 3-\frac{x}{2}\right]$ we have $-y^2+y(6-x) \geq$
$\min\left\{-x^2+x(6-x), -\left(3-\frac{x}{2}\right)^2+\left(3-\frac{x}{2}\right)(6-x)\right\} \geq -x^2+x(6-x) = 2x(3-x)$
because $-\left(3-\frac{x}{2}\right)^2+\left(3-\frac{x}{2}\right)(6-x)+x^2-x(6-x) = \frac{9}{4}(x-2)^2.$

Hence, for any $x \in [1,2]$ we have
$xyz-4 = x(-y^2+y(6-x))-4 \geq 2x^2(3-x)-4 = 2(x^2(3-x)-2) =$
$2(x-1)(2+2x-x^2) = 2(x-1)\left(3-(x-1)^2\right) \geq 2(x-1)\left(3-(2-1)^2\right) = x-1 \geq 0.$

Since lower bound 4 is attained if $(x,y,z) = (1,1,4)$ then $\min xyz = 4.$

Thus, $q \geq q_* := \max\left\{4, \frac{8p-72}{3}\right\}$ and
$216-\left((12-q)^2+2(p-2)^2\right) \geq 216-\left((12-q_*)^2+2(p-2)^2\right).$

Since $3 < p = xy+yz+zx \leq \frac{(x+y+z)^2}{3} = \frac{36}{3} = 12$, $q_* = 4$ if $\frac{8p-72}{3} \leq 4 \Leftrightarrow 3 < p \leq 21/$
$q_* = \frac{8p-72}{3}$ if $21/2 \leq p \leq 12$ then:

For $p \in [21/2, 12]$ we have
$$216-\left((12-q_*)^2+2(p-2)^2\right) = 216-\left(\left(12-\frac{8p-72}{3}\right)^2+2(p-2)^2\right) =$$
$$\frac{2(41p-408)(12-p)}{9} \geq \frac{2(41\cdot(21/2)-408)(12-p)}{9} = 5(12-p) \geq 0;$$

For $6\sqrt{2} \leq p \leq 21/2$ we have
$$216-\left((12-q_*)^2+2(p-2)^2\right) = 216-\left((12-4)^2+2(p-2)^2\right) =$$
$2(72+4p-p^2) \geq 0$ because $72+4p-p^2 = 76-(p-2)^2 \geq 76-(21/2-2)^2 = \frac{15}{4} > 0.$